

Micro-422: Lasers: theory and modern applications

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Homework 6: Shot noise, Johnson noise, Heterodyne detection

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1. The Mach-Zehnder Amplitude modulator

A Mach Zehnder modulator is used to achieve amplitude modulation of light as required for transmitting data over an optical fiber. In this device the input light is split equally into two waveguide modes, which are at the end of the device joined again into a single waveguide mode. Next to each of the waveguides there is an electrode which creates a homogeneous electric field E over a total length L . The substrate is made from an electro-optic material (such as LiNbO_3) which changes its refractive index, when an Electric field is applied. The change of refractive index is given by: $\delta n = E \cdot r \cdot n^3 / 2$ where the refractive index $n = 1.8$ is the optical refractive index of LiNbO_3 and $r = 30 \times 10^{-12} \text{ m V}^{-1}$ is the electro-optical coefficient. Assume that the waveguide length is $L = 5 \text{ mm}$ and that the separation of the electrode pairs is $s = 20 \mu\text{m}$ and that the dielectric constant of LiNbO_3 at the modulation frequency (typically GHz) is $\epsilon = 20$. Assume that the electric field between the electrodes is homogeneous. Note that electric field on the two waveguides is opposite, such that the induced phase shift takes on opposite sign in each of the waveguides. The height of the electrodes is $10 \mu\text{m}$

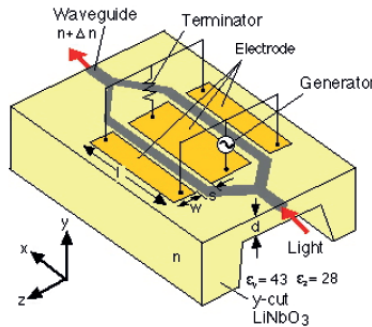


Figure 1: Schematics of an integrated Mach-Zehnder Amplitude Modulator.

- Derive an expression for the v_π , i.e. the voltage that is needed to shift the two arms by 180 degrees (π), which is required to cause destructive interference. Derive the value for the modulator with the above parameters.
- Derive the fastest speed at which you could modulate the light using this device for $\lambda = 1550 \text{ nm}$

2. Photons

A laser pointer emits 3 mW of power at 532 nm.

- How many photons, i.e. quanta of light, are emitted per second?

- (b) Assuming that you have a photodetector that has a unit conversion efficiency (i.e. converts one photon into a single electron) what would be the current that would be generated in the photodiode?
- (c) Suppose that you are seated 10m from a screen onto which a speaker shines his laser pointer. Assume that the screen scatters 20 % of the photons isotropically into the room. Estimate the number of photons per second that enter your eye, when the eye pupil has a diameter of 5 mm.

3. The Poisson distribution

The Poisson distribution describes the statistical properties of random processes and can be found in a variety of processes in nature, ranging from radioactive decay to defects in Genes. If during the observation time T , the average number of recorded events is $\bar{n} = \langle n \rangle$ then the probability distribution to detect in such an interval n events is given by the Poisson distribution:

$$p(n) = \frac{(\bar{n})^n}{n!} e^{-\bar{n}} \quad (10)$$

where $\bar{n}$ is the average number of detection events.

- (a) Show that the Poisson distribution is normalized, i.e. $\sum_n p(n) = 1$
- (b) Show that the average of the Poisson distribution is given by $\bar{n}$.
- (c) Show that the variance, as defined by $\text{Var}[n] = \langle (n - \bar{n})^2 \rangle$ is given by: $\text{Var}[n] = \bar{n}$.

4. Shot noise

Electronic shot noise in optical detection is a consequence of the fact that the current is composed of electrons with discrete charges. It can equally be understood by noting the electromagnetic field is quantized. This noise is fundamental and independent of the type of detector, its material, or temperature. Assume that the photoelectric detection is also a process that exhibits a Poisson distribution. Show that the fluctuations in the photocurrent in a measurement time T are given by

$$\text{Var}[i] = i_{\text{var}} = e \cdot \bar{i} \cdot \frac{1}{T} \quad (20)$$

where $\bar{i}$ is the average photo-current.¹

5. Shot noise limited detection

We consider a photodiode that has a quantum efficiency of 90 % to a load resistance of $R = 50\text{ohm}$ (see figure). Assume that 1mW of optical power is incident on the detector and that the laser transmits music, which you would like to detect and hear (the highest tone we can perceive is ca. 4 kHz). Assume that the modulated power is converted to an average photo-current of $\bar{i} = P \cdot e / \hbar \omega \cdot \eta$. This quantity is known as the responsivity ($R = e / \hbar \omega \cdot \eta$). Assume that the detector has a measurement bandwidth of 4 kHz, the minimum which is required to measure the signal modulation.

- (a) Calculate the fluctuations in the photocurrent (i.e. the root-mean-square photocurrent), when the photocurrent is measured over an interval of $2 \cdot T = 0.25 \text{ kHz}$.
- (b) What is the signal to noise in this case? ²

¹Note that because the total root-mean-square current fluctuations depend on the measurement time, it is useful to introduce the spectral density, $S_i(\nu) = \langle \Delta i^2 \rangle / B = 2e\bar{i}$ where B is the bandwidth ($1/2T$). This spectral density is white, i.e. frequency independent.

²The SNR is given by the $SNR = \frac{\bar{I}^2}{\text{Var}[I]}$ where I is the signal photo-current and $\text{Var}[I]$ is the root mean square fluctuations of the photocurrent.

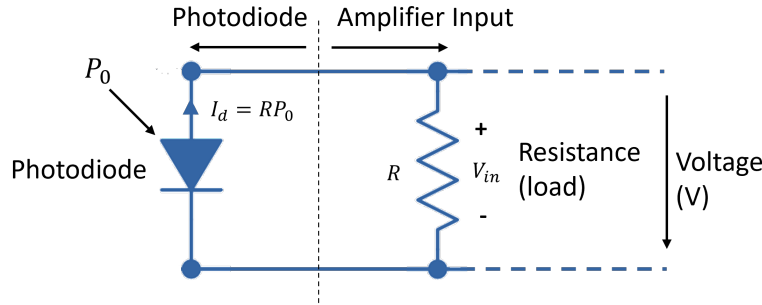


Figure 2: Photodiode and resistance circuit.

- (c) What is the lowest power incident on the detector, such that you can still hear the music (that is, have an SNR of unity)?
- (d) What happens to the signal-to-noise ratio when you use instead of a 4 kHz sampling bandwidth, a sampling rate of the photocurrent that is 4 GHz, i.e. $\times 1000$ times faster?

6. Thermal noise

The last exercise should have shown you that one can obtain a very large signal to noise in optical detection. One of the problems in optical detection however is that it is in practice difficult to realize this signal to noise due to thermal noise. This noise originates from the fact that the carriers in a resistor are thermally agitated and fluctuating (at temperature T). Equivalently this implies that using Ohm's Law there are random current fluctuations. The fluctuations are given by:

$$\text{Var}[i] = 4 \cdot k_B \cdot T / R \cdot B = S_{TN}(\nu) B \quad (33)$$

- (a) Calculate the thermal Johnson noise in a 50Ω resistor at room temperature (300 K) in a measurement bandwidth of 10 kHz.
- (b) Assume that the photodetector is connected the resistance, which produces an average photocurrent but in addition also a fluctuating current due to the shot noise. Write down an expression for the signal-to-noise ratio (SNR) for the voltage fluctuations that drops across the resistance (R) shown in the adjacent figure. Note that the two noises are uncorrelated, such that the total noise on the resistance will be the sum of the variances.
- (c) What power would you need for an ideal detector to make the shot noise fluctuations as large as those from the thermal noise if the bandwidth of detection is 10 kHz?

7. Design of fiber optical cable for data transmission

You are responsible for designing a custom fiber link between the trading floor at the Chicago commodity exchange and a bank in a nearby city, 100 km away. The link is designed to help the bank in ultrafast trading.

- (a) You are asked to specify the required laser power to enable efficient communication. The trading floor is sending data at a rate of 1×10^9 bits/s (which implies that the detector bandwidth needs to be at least 1/period of one bit, which is $B = 1 \times 10^9$ Hz). In addition, the system is supposed to operate with a signal-to-noise of 22 dB ³. Assume that the fiber has an attenuation coefficient

³Which implies that only 1 bit in 10^{10} is incorrectly detected due to the noise.

of 0.5 dB km^{-1} for the used wavelength of $\lambda = 1550 \text{ nm}$. The detector that is available for the transmission experiment is terminated into a $R = 50 \Omega$ load. Assume the detector is at room temperature ($T = 300 \text{ K}$).

8. Heterodyne detection

In the case of very weak signals, it is difficult to maintain a high signal-to-noise ratio (SNR) due to the presence of thermal (Johnson) noise. One way to overcome this obstacle would be to amplify the signal with an optical fiber amplifier. This is however not always possible. A second technique to measure very small signals is to employ a heterodyne detection in which the signal is superimposed with a local oscillator (PLO) which is at a different frequency ω_{LO} (the difference being $\Omega = \omega_S - \omega_{LO}$). Assume that the local oscillator is much stronger in strength than the signal.

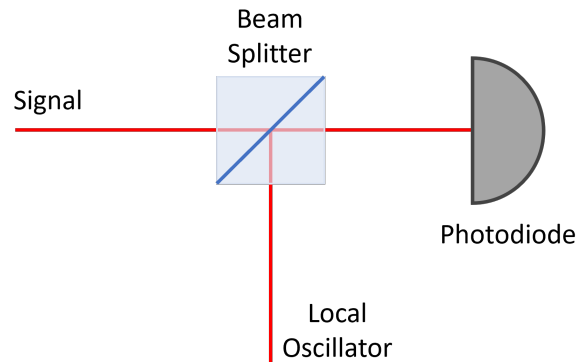


Figure 3: Basic idea of heterodyne detection.

- (a) Show that the absolute magnitude of the photocurrent produced by the signal in the heterodyne case is higher than in the case of direct detection.
- (b) Calculate the signal-to-noise ratio (SNR) for the case of a heterodyne detector, taking into account shot noise caused by the local oscillator average current.
- (c) You would like to detect a signal of only 100 nW . What is the required power in the LO in order to overcome the thermal noise if your measurement bandwidth is given by 10 MHz ? We assume a resistance of 50Ω and $T = 300 \text{ K}$